The Rise Velocity of Bubbles in Tubes and Rectangular Channels As Predicted by Wave Theory

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In a previous paper, an analogy was presented which related the rise velocity of bubbles in liquids of infinite extent to the velocity of surface waves over infinitly deep liquids. In the present paper, a corresponding analogy is found which relates bubble velocities in bounded liquids to wave velocities over liquids of finite depth.

Except for certain limiting cases, the general problem of predicting the rise velocity of individual bubbles in restricted media has not yielded to theoretical treatment. A major part of the problem is related to the mathematical difficulties of expressing the complex interaction of bubble shape and flow field. The problem is further compounded when wall effects are introduced.

Previous attempts to analyze the wall effect have sought to derive an analogy between the dynamics of bubbles and rigid spheres. An empirical equation based on such an analogy has been obtained by Uno and Kintner (*1*). Their correlation, however, suffers from a serious theoretical deficiency in that it predicts a zero value for the terminal velocity of bubbles (slugs) which are large enough to fill the tube. It is known, however, that except for tubes of very small diameter, slugs have a finite rise velocity which can be quite accurately predicted by an equation of the form derived by Dumitrescu (*3*) or Davies and Taylor (*4*).

While it is no doubt possible to modify the rigid sphere analogy to include the limiting case of slugs (*2*), it is questionable whether the theoretical basis for the analogy is in itself justified. It would seem that the basic problem of flow induced deformations of bubble shape cannot possibly be reflected in the motion of a rigid particle. This contention is supported by the work of Harmathy (*2*), in which the terminal velocities of solid spheres and of gas bubbles in tubes were compared and shown to be quite dissimilar. This is particularly true at the two limiting conditions of small particles (approximately 0.4 cm. in diameter) in infinite media and particles contained in tubes of comparable diameter. In the former instance, solid spheres tend to move rectilinearly, whereas bubbles exhibit a helical or zigzag motion. In the latter case, the terminal velocity of spheres is ultimately reduced to zero as the sphere fills the tube. Bubbles, on the other hand, deform to the slug shape and approach a limiting but nonzero velocity.

It is the purpose of this paper, therefore, to present an alternate, although still heuristic, approach to the problem which may be more tenable. This approach is based on an analogy which was found to exist (*5*) between the dynamics of bubbles rising in infinite media and the propagation of surface waves over deep liquids. The analogy has been extended in the present paper to include bubbles rising in restricted media by assuming that the presence of solid boundaries affects bubble motion and wave motion in a dynamically similar manner.

First, it will be shown how the analogy may be applied to three-dimensional bubbles rising in cylindrical tubes. The methods developed for three-dimensional bubbles will then be used to obtain the rise velocity of two-dimensional or plane bubbles. Bubbles rising between infinite parallel plates and in rectangular channels of finite width will be treated. These cases represent, respectively, the infinite media and restricted media analogues in two dimensions.

PREVIOUS WORK

The works of Dumitrescu and Davies and Taylor are the progenitors of much of the experimentation and analysis dealing with bubble rise velocities. These authors proposed that the behavior of bubbles rising through actual liquids could be approximated by the hypothetical case of a bubble rising through an inviscid liquid when surface tension effects are unimportant. With the additional assumption of constant pressure within the bubble, Davies and Taylor proceeded to obtain the velocity which satisfied Bernoulli's equation at the frontal stagnation point. This gave the velocity of a spherical cap bubble in an infinite liquid as

$$U_\infty = \frac{2}{3} (ga)^{1/2} \quad (1)$$

where a is the frontal radius of curvature.

Collins (*6*), using the same approach, analyzed the rise velocity of plane bubbles between infinitely wide parallel plates and found that

$$U_\infty = 0.5 \, (ga)^{1/2} \quad (2)$$

The rise velocity of an infinitely long bubble or slug rising in a tube was also given by Davies and Taylor; however, Dumitrescu's analysis is considered the better approximation. He obtained the potential function for the flow in a tube and assumed a spherical nose for the bubble. Solving simultaneously the flow around the spherical nose and the asymptotic film flow, he obtained the bubble velocity and the frontal radius of curvature. The asymptotic film profile was obtained from continuity by assuming a constant velocity in the film. The result obtained for the rise velocity was

$$U_s = 0.495(gr)^{1/2} \quad (3)$$

Garabedian (*7*) discussed the equivalent plane case and showed that for a channel of half-width b

$$U_s = 0.337(gb)^{1/2} \quad (4)$$

while Birkhoff and Carter (*8*) obtained a constant of 0.33 ± .01. Equations (1) through (4) are important in that they are the asymptotes of the general solution of bubbles rising in circular and rectangular channels.

Uno and Kintner (*1*) experimentally investigated the behavior of bubbles rising in tubes. A review of the literature on solid spheres led these authors to conclude that the presence of the tube wall could be accounted for by the general equation

$$\frac{U}{U_\infty} = \Phi \left(1 - \frac{d}{D} \right)^n \quad (5)$$

By assuming that Equation (5) would apply equally well to bubbles, they were able to correlate their data and evaluate n and Φ to obtain the relation

$$\frac{U}{U_\infty} = \left[\frac{1}{B} \left(1 - \frac{d_e}{D} \right) \right]^{0.765} \quad (6)$$

where B is a factor which depends on tube diameter and surface tension. This dependence was presented in graphical form in the original paper and is reproduced in Figure 1 for later reference.

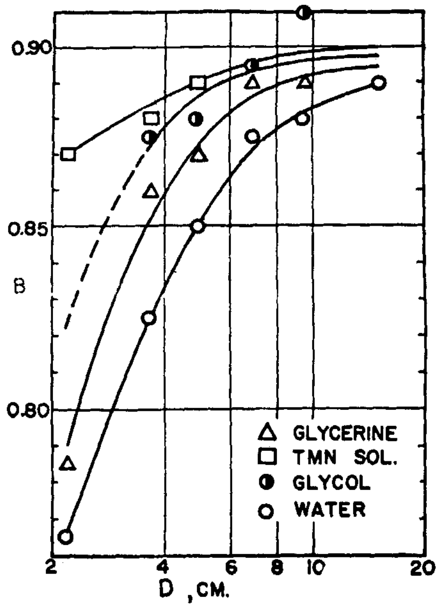


Fig. 1. Values of *B* as a function of tube size and surface tension (from reference 1).

As noted previously, Equation (6) cannot be used for bubbles which are large compared with the tube diameter. Uno and Kintner noted this deficiency and restricted the equation to values of $(d_e/D) < 0.5$.

Harmathy improved the above correlation by noting that the surface tension effect could be expressed in terms of the Eötvös number ($N_{E\ddot{o}} = g\rho r^2/\sigma$). However, in order to encompass all values of d_e/D, he found it necessary to develop a two-region correlation, each region valid for a specific range of d_e/D.

Collins (*6*) derived an approximate analytical expression for the equivalent two-dimensional case of plane bubbles rising in rectangular channels. He showed that it gave reasonably good agreement with experiment, although it could not account for the three-dimensional effects due to the finite channel spacing.

In a recent analysis, Mendelson (*5*) showed that a direct analogy exists between the propagation velocity of surface waves on a deep liquid and the rise velocity of bubbles in infinite media. For deep inviscid liquids, the wave velocity is given by (*9*)

$$C_\infty = \left(\frac{2\pi\sigma}{\lambda\rho} + \frac{g\lambda}{2\pi} \right)^{1/2} \quad (7)$$

It was noted that the above equation exhibits many of the properties required to characterize the terminal velocity of bubbles. This similarity was found to be quantitative when the perimeter of the maximum bubble cross section $2\pi r_e$ was substituted for the wavelength λ. The equivalent radius r_e is the radius of a sphere whose volume is equal to that of the bubble. The resulting equation

$$U_\infty = \left(\frac{\sigma}{\rho r_e} + g r_e \right)^{1/2} \quad (8)$$

was found to correlate the data for bubbles rising in pure liquids. A theoretical understanding of the mechanism by which the perimeter of the equivalent sphere affects the rise velocity is not yet clear. However, the usefulness of the perimeter concept and its further interpretation as the perimeter normal to the direction of motion will be demonstrated when the analogy is applied to a two-dimensional geometry.

BUBBLES RISING IN TUBES

It is known (*1*) that bubbles rise slower in bounded liquids than they do in liquids of infinite extent. Waves also exhibit this property and travel slower in shallow liquids than they do in deep liquids. To obtain the analogy between the two phenomena, a more general form of the wave equation, which accounts for the effects of liquid depth, will be utilized (*9*); thus

$$C = \left[\left(\frac{2\pi\sigma}{\lambda\rho} + \frac{g\lambda}{2\pi} \right) \tanh \frac{2\pi h}{\lambda} \right]^{1/2} \tag{9}$$

When the equivalent perimeter is substituted for the wavelength as before, Equation (9) becomes

$$U = \left[\left(\frac{\sigma}{pr_e} + gr_e \right) \tanh \frac{h}{r_e} \right]^{1/2} \tag{10}$$

Equations (8) and (10) are combined to give

$$\frac{U}{U_\infty} = \left(\tanh \frac{h}{r_e} \right)^{1/2} \tag{11}$$

In the above equation, the parameter h is as yet undefined. However, by analogy, it should be related to some effective liquid depth which is determined by the geometry of the system boundaries. For the case of bubbles rising in tubes, h should be a function of the tube radius. The simplest such function is one of direct proportionality:

$$h = C_1 r \tag{12}$$

Substitution of Equation (12) into (11) yields the relatively simple expression

$$\frac{U}{U_\infty} = \left[\tanh C_1 \frac{r}{r_e} \right]^{1/2} \tag{13}$$

The constant C_1 is to be determined from the known rise velocity of slugs. This rise velocity is given by the more general form of Dumitrescu's equation

$$U_s = C_2 (gr)^{1/2} \tag{14}$$

where C_2 is a complex function of viscosity and surface tension (*10, 11*).

For the various liquids and tube diameters investigated by Uno and Kintner, it can be shown that C_2 has an approximately constant value of 0.495. By assuming that the slug rise velocity is attained when $r_e = r$, Equations (8), (13), and (14) may be combined to give

$$\frac{U_s}{U_\infty} = \frac{0.495}{[(1/N_{E\ddot{o}}) + 1]^{1/2}} = (\tanh C_1)^{1/2} \tag{15}$$

For the moment, $N_{E\ddot{o}}$ will be restricted to large values, that is, large tube diameters. This permits C_1 to be easily evaluated; it is found to be 0.25, so that Equation (13) becomes

$$\frac{U}{U_\infty} = \left[\tanh 0.25 \frac{r}{r_e} \right]^{1/2} \qquad N_{E\ddot{o}} \to \infty \tag{16}$$

Equation (16) represents the simplest expression which can be derived from the wave analogy. At first glance, this equation seems to be quite unrelated to the equation proposed by Uno and Kintner (*6*). However, the numerical comparison, presented below, shows that the equations are actually in very good agreement.

Comparison with data

Large Eötvös Numbers. The data in Figure 1 indicate that for large tube diameters, the parameter B tends toward a value of 0.9 for all of the fluids tested. With this value of B, Uno and Kintner's correlation may be compared with the wave analogy, Equation (16). This comparison is shown in Figure 2. It can be seen that in the range in which Equation (6) is reported to apply, that is, $r/r_e > 2.0$, the agreement is within 1%. In addition, Equation (16) extends the correlation to large bubble sizes. This will be shown presently by comparison with Uno and Kintner's actual data.

Small Eötvös Numbers. The rise velocity of both bubbles and slugs are influenced by surface tension effects at low Eötvös numbers. White and Beardmore (*10*) have shown that the terminal velocity of slugs decrease with decreasing Eötvös number. Wallis (*11*) obtained an empirical correlation for this effect which, for low viscosity fluids, has the form

$$C_2 = 0.495 \left(1 - e^{\frac{3.37 - 4N_{E\ddot{o}}}{10}} \right) \tag{17}$$

where C_2 is the parameter referred to in Equation (14). On the other hand, Uno and Kintner's data indicate that the terminal velocity of bubbles increase at low Eötvös numbers.

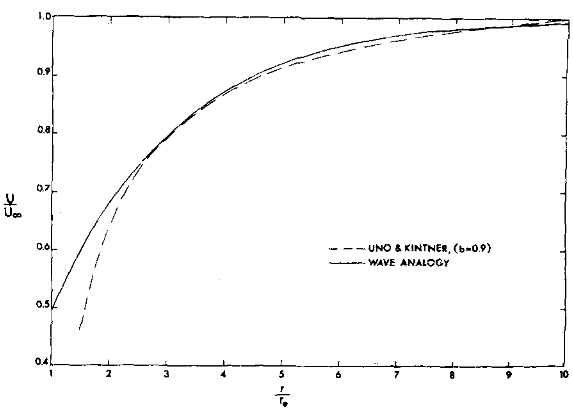


Fig. 2. Comparison of Uno and Kintner's correlation and the wave analogy for large Eotvos numbers.

The first effect occurs when slugs are generated in a small diameter tube containing a high surface tension liquid. The curvature at the nose of the slug is then distorted to such an extent that the free flow of liquid past the slug is reduced or prevented.

The latter effect is apparently due to the helical trajectory of small bubbles which cause them to rise at some mean position other than the tube axis. It seems reasonable to expect that this axial displacement of the bubble would have a more pronounced affect on the rise velocity in small diameter tubes rather than in large ones. It will be shown, later in this paper, that the rise velocity of plane bubbles in rectangular channels does not depend on the Eötvös number effect. This is understandable, however, since plane bubbles are prevented from spiraling by the channel walls.

Both of the above surface tension effects apparently have no analogue in classical, small amplitude, wave theory. While it may be possible to find a correspondence to these effects in the more complicated nonlinear wave theories, no attempt will be made to do so in the present paper. Instead, an empirical corrective term will be added to the effective liquid depth to account for bubble spiraling, and, in addition, C_1 will be allowed to vary in accordance with the Wallis correlation, Equation (17).

Thus, Equation (12) is modified as follows:

$$h = C_1 r + \frac{3.0}{N_{E\ddot{o}}}(r - r_e) \tag{18}$$

Substitution of the above into Equation (11) yields

$$\frac{U}{U_\infty} = \left\{\tanh\left[C_1\frac{r}{r_e} + \frac{3.0}{N_{E\ddot{o}}}\left(\frac{r}{r_e} - 1\right)\right]\right\}^{1/2} \tag{19}$$

As before, the value of C_1 is determined from the limiting slug velocity. When one utilizes Wallis' correlation at $r_e = r$, the defining equation for C_1 is

$$C_1 = \tanh^{-1}\left[\frac{0.245\left(1 - e^{\frac{3.37 - 4N_{E\ddot{o}}}{10}}\right)^2}{(1/N_{E\ddot{o}}) + 1}\right] \tag{20}$$

Since the maximum value of C_1 is 0.25, Equation (20) can be approximated by

$$C_1 \approx \frac{0.245\,N_{E\ddot{o}}\left(1 - e^{\frac{3.37 - 4N_{E\ddot{o}}}{10}}\right)^2}{1 + N_{E\ddot{o}}} \tag{21}$$

Equation (19) is tested against Uno and Kintner's data in Figures 3 to 6. In general, the agreement is within the experimental scatter.

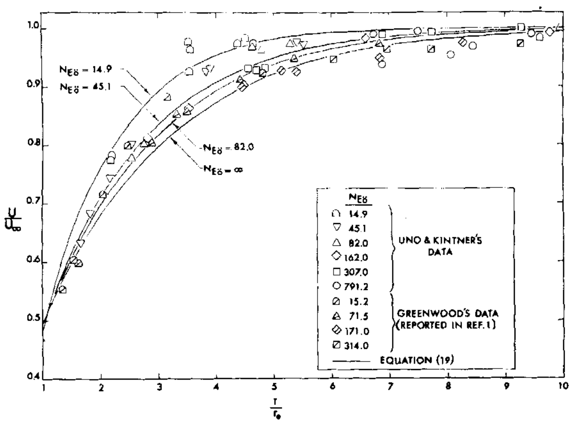


Fig. 3. Rise velocity of bubbles in tubes containing distilled water.

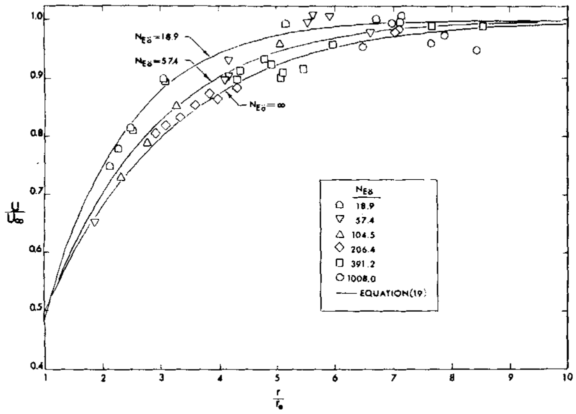


Fig. 4. Rise velocity of bubbles in tubes containing a 61% glycerine solution.

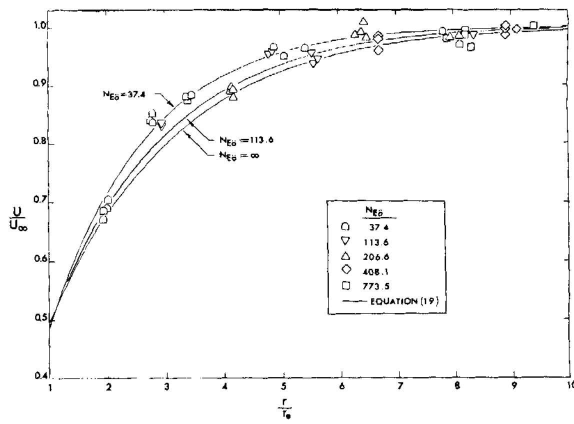


Fig. 5. Rise velocity of bubbles in tubes containing trimethyl nonyl ether solution.

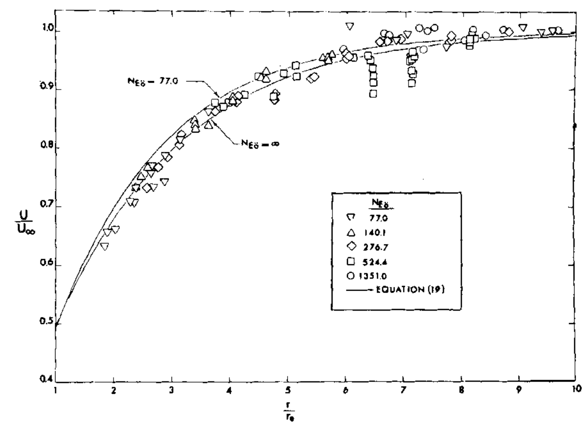


Fig. 6. Rise velocity of bubbles in tubes containing diethylene glycol.

BUBBLES RISING BETWEEN INFINITE PARALLEL PLATES

The available area for liquid flow past a plane bubble rising between infinite plates is unlimited. In this respect, the infinite plate geometry can be considered as the two-dimensional analogue of the three-dimensional infinite media case considered above.

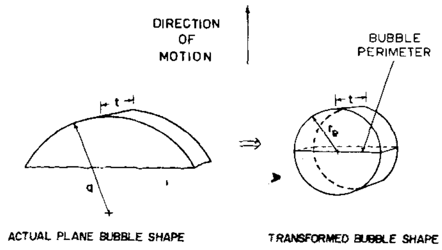
In order to extend Equation (7) to plane bubbles it is necessary to obtain the appropriate bubble perimeter. In the three-dimensional case, the bubble was first transformed into a sphere of equivalent volume. The perimeter was then taken as the circumference of the maximum cross section.

By analogy, the plane bubble is first transformed into a cylinder of equal volume. This transformation is illustrated in Figure 7. The perimeter is now taken as that of the maximum area normal to the direction of motion; thus

$$p = 4r_e + 2t \tag{22}$$

where r_e is given by

$$r_e = \left(\frac{V_b}{\pi t}\right)^{1/2} \tag{23}$$



$V_b = \pi r_e^2 t$

Fig. 7. Definition of equivalent radius and perimeter for plane bubble.

and V_b is the volume of the bubble. As in the three-dimensional case, the perimeter is set equal to the wavelength. Substitution of Equation (22) into Equation (7) yields

$$U_\infty = \left[\frac{\pi\sigma}{(2r_e + t)\rho} + \frac{g(2r_e + t)}{\pi} \right]^{1/2} \tag{24}$$

or

$$U_\infty = \left[\frac{\pi\sigma}{r_e(2 + \gamma)\rho} + \frac{g r_e(2 + \gamma)}{\pi} \right]^{1/2} \tag{25}$$

where $\gamma = t/r_e$. For large bubble volumes and small spacings $\gamma \to 0$ and the surface tension term becomes negligible, so that Equation (25) becomes

$$U_\infty = \left(\frac{2 g r_e}{\pi} \right)^{1/2} \tag{26}$$

In its present form, Equation (26) is not directly comparable to existing results which are expressed in terms of the frontal radius of curvature of the plane bubble. Collins, however, has found experimentally that plane bubbles rising between large parallel plates are approximately segments of a circle and attain a limiting included angle of about 105 deg. With this information it is easy to show that

$$r_e = \beta a \tag{27}$$

where $\beta = 0.371$ (see Appendix A).

Elimination of r_e from Equation (26) yields

$$U_\infty = 0.486\,(ga)^{1/2} \tag{28}$$

This result agrees well with Equation (2), the coefficients differing by 2.8%. When experimental scatter for the limiting included angle is taken into account, a value as high as 0.509 is obtained for the coefficient in Equation (28).

BUBBLES RISING IN RECTANGULAR CHANNELS

For this case the analogy is made with the propagation velocity of waves on liquids of finite depth. Because the data of Collins was obtained for large bubbles, surface tension may be ignored, and Equation (9) becomes

$$U = \left[\left(\frac{g\lambda}{2\pi} \right) \tanh \frac{2\pi h}{\lambda} \right]^{1/2} \tag{29}$$

The wavelength λ is replaced with the bubble perimeter obtained earlier [Equation (22)] to yield

$$U = \left[\left(\frac{g r_e(2 + \gamma)}{\pi} \right) \tanh \frac{\pi h}{r_e(2 + \gamma)} \right]^{1/2} \tag{30}$$

Again, when one follows the approach developed for three-dimensional bubbles, it will be assumed that h is some fraction of the half-width of the channel, that is

$$h = C_3 b \tag{31}$$

and that the limiting slug velocity is given by Griffith's (12) experimental relation for rectangular channels

$$U_s = (0.23 + 0.065\gamma_o)(2gb)^{1/2} \tag{32}$$

where $\gamma_o = t/r_e\,_{r_e=b}$. In addition, it will also be assumed that a slug first forms when $r_e = b$. Under these assumptions Equation (30) can be rewritten as

$$U_s = \left[\left(\frac{gb(2 + \gamma_o)}{\pi} \right) \tanh \frac{\pi C_3}{2 + \gamma_o} \right]^{1/2} \tag{33}$$

The quantity C_3 is then found to be

$$C_3 = \left(\frac{2 + \gamma_o}{\pi} \right) \tanh^{-1}\left[\frac{2\pi(0.23 + 0.065\gamma_o)^2}{2 + \gamma_o} \right] \tag{34}$$

With the aid of the relation $\gamma_o = \gamma r_e/b$, Equation (34) can be written in the more convenient form

$$C_3 = \left(\frac{2 + (\gamma r_e/b)}{\pi} \right) \tanh^{-1} \left\{ \frac{2\pi\,[0.23 + (0.065\gamma r_e/b)]^2}{2 + (\gamma r_e/b)} \right\} \tag{35}$$

If Equation (30) is divided by the velocity of a bubble of equal volume rising between infinite parallel plates [Equation (25) with surface tension neglected], the result is

$$\frac{U}{U_\infty} = \left[\tanh \frac{\pi C_3 (b/r_e)}{2 + \gamma} \right]^{1/2} \tag{36}$$

When γ is negligible, the quantity C_3 takes on the constant value 0.107, and the normalized rise velocity becomes simply

$$\frac{U}{U_\infty} = \left(\tanh 0.1675 \frac{b}{r_e} \right)^{1/2} \tag{37}$$

Comparison with experiment

To the authors' knowledge the only data taken on plane bubbles is that of Collins. His experimental apparatus consisted of two 3 ft. square plastic plates spaced ¼ in. apart with movable side walls. The width of the channel was varied from 33 to 2.5 in. At the maximum width the minimum sized bubble had a radius of curvature of 1.5 in. which gives a value of b/r_e of 29.3. Substitution of this value into Equation (37) gives $U \approx U_\infty$, so that the bubble behaves as if it were rising between infinite parallel plates. The rise velocity obtained by Collins for this case was

$$U_\infty = 0.545\,(ga)^{1/2} \tag{38}$$

which is 9% higher than the theoretical value [Equation (2)]. Collins attributed this discrepancy to three-dimensional effects. It will be shown that the choice of bubble perimeter [Equation (22)] accounts for this effect. With the aid of Equation (27), Equation (25), with surface tension neglected, can be rewritten as

$$U_\infty = \left[\frac{ga\,(2\beta + t/a)}{\pi} \right]^{1/2} \tag{39}$$

When the value of a for the minimum sized bubble is substituted along with the corresponding value of β obtained from Figure 8, Equation (39) becomes

$$U_\infty = 0.538\,(ga)^{1/2} \tag{40}$$

This result differs by only 1.3% from Collins experimental value.

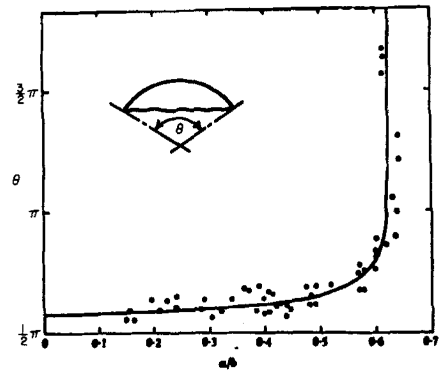


Fig. 8. Variation of measured included angle.

The analytical result obtained for rectangular channels [Equation (36)] is compared with Collin's data in Figure 9 for two values of the parameter γ. The curves are to be interpreted as the results of an experiment in which the bubble volume and channel spacing are held constant while the channel width is varied. The upper curve corresponds to bubbles of large equivalent radius ($r_e >> t$, that is, $\gamma \approx 0$), while the lower one corresponds to $\gamma = 0.45$ which is the maximum value obtained by Collins. It is seen that the two curves essentially bracket the data which represent various values of γ between 0 and 0.45.

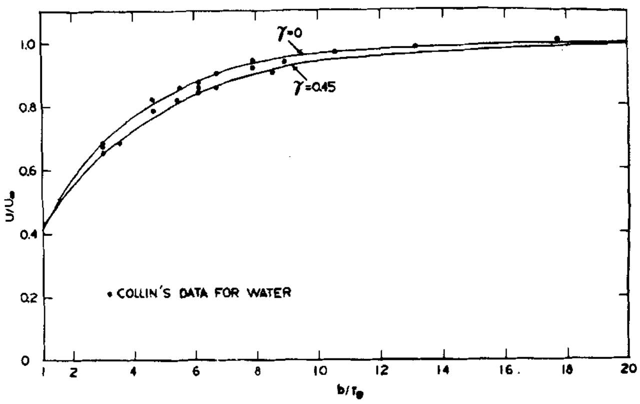


Fig. 9. Variation of bubble velocity.

CONCLUSIONS

It has been shown that surface wave theory can be used to describe the motion of bubbles rising in tubes and channels when the following similitude relationships are used:

1. The wavelength is equivalent to the perimeter of the bubble cross section normal to the direction of motion, the perimeter being based upon an equivalent radius previously defined.
2. The liquid depth of wave theory is equivalent to a calculable fraction of the tube radius or channel width.

NOTATION

a = frontal radius of curvature of bubble, cm.
b = half-width of rectangular channel, cm.
B = function of surface tension and tube diameter, dimensionless
C = phase velocity of surface waves on liquids of shallow depth, cm./sec.
C_∞ = phase velocity of surface waves on liquids of infinite depth, cm./sec.
C_1 = proportionality constant in Equation (12), dimensionless
C_2 = function defined by Equation (17), dimensionless
C_3 = proportionality constant in Equation (31), dimensionless
d = diameter of solid sphere, cm.
d_e = $2r_e$, cm.
D = $2r$, cm.
g = gravitational constant, 980 cm./sec.^{2}
h = depth of liquid in wave theory, cm.
$N_{E\ddot{o}}$ = Eötvös number based on tube radius $g\rho r^2/\sigma$, dimensionless
P = perimeter of maximum cross section, normal to motion, of transformed bubble, cm.
r = tube radius, cm.
r_e = radius of sphere or cylinder having the same volume as the three dimensional or two-dimensional bubble, respectively, cm.
t = channel thickness, cm.
U = rise velocity of bubble in bounded liquids, cm./sec.
U_∞ = rise velocity of bubble in infinite media, cm./sec.
U_s = rise velocity of slug, cm./sec.
V_b = bubble volume, cc.
λ = wavelength, cm.
ρ = liquid density, g./cc.
σ = surface tension, dynes/cm.

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APPENDIX A

Area of Circular Segment

The area of a circular segment is given by the relation

$$Acs = \frac{a^2\,(\theta - \sin\theta)}{2} \tag{A1}$$

where a is the radius of the circle and θ is the included angle of the segment. The radius of a circle having the same area is then

$$r_e = a\left[\frac{\theta - \sin\theta}{2\pi}\right]^{1/2} \tag{A2}$$

or

$$r_e = \beta a \tag{A3}$$

Values for β can be obtained from the data of Collins given in Figure 8. Photographs taken by Collins showed that the bubbles were circular segments up to a value of $a/b = 0.6$.